REEL-C 4519

A.T.I 93359

STABILITY OF THE SUPERSONIC LAMINAR BOUNDARY LAYER WITH A PRESSURE GRADIENT

By

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Report No. 167

November 20, 1950

UNITED STATES NAVY

Office of Naval Research
Mechanics Branch, Mathematical Sciences Division

and

UNITED STATES AIR FORCE

Office of Air Research

LEGIBILITY POOR

(Contract N6-ori-270, Task Order #6,

Project Number NR 061-049)

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WITH A PRESSURE GRADIENT

UNCLASSIFIED

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PRINCETON UNIVERSITY, AERONAUTICAL ENGINEERING LAB., N.J.
(REPORT NO. 167)

STABILITY OF THE SUPERSONIC LAMINAR BOUNDARY LAYER WITH A
PRESSURE GRADIENT - AND APPENDIX A

LESTER LEES 20 NOV'50 39PP TABLES, GRAPHS

OFFICE OF NAVAL RESEARCH, WASH., D.C., USN CONTR. NO.
N6-ORI-270

AERODYNAMICS (2) BOUNDARY LAYER, LAMINAR - STABILITY
BOUNDARY LAYER (5) FLOW, SUPERSONIC

UNCLASSIFIED

PRINCETON UNIVERSITY

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SUMMARY

Qualitative considerations concerning the effect of negative pressure gradient on the stability of the supersonic laminar boundary layer are supplemented in this paper by calculations of the approximate minimum critical Reynolds number, or stability limit, for the boundary layer on several insulated, supersonic, symmetrical circular-arc airfoils at Mach numbers of 1.5, 2.0, and 3.0. Velocity and temperature distributions across the boundary layer are obtained by the Dorodnitzyn-Pohlhausen method based on the von Kármán momentum integral equation.

At low supersonic Mach numbers the laminar boundary layer over an insulated surface is completely stabilized when the modified Pohlhausen parameter $\lambda \sim \frac{\delta^2}{\nu} \frac{dU}{dx} f(M)$ is larger than a certain critical value that depends only on the Mach number and the properties of the gas. For example, at $M = 1.5$, $\lambda_{cr} \cong 6.5$; at $M = 1.75$, $\lambda_{cr} \cong 7$, and at $M_1 = 2.0$ $\lambda_{cr} \cong 11$. For $M_1 = 2.0$, the destabilizing effect of aerodynamic heating is dominant and the increase of the minimum critical Reynolds number, $Re_{\delta_{cr_{min}}}$ with λ is small until $\lambda \to 11$. For $M_1 = 3.0$, the influence of negative pressure gradient is negligible, at least for an insulated surface.

Comparisons between the calculated distribution of the stability limits over the airfoil at $M_1 = 1.5$ and the growth of the Reynolds number based on the boundary layer thickness for various flight Reynolds numbers (Re_c) show that,

a) For the six per cent thick airfoil at $\alpha = 0^\circ$ a region of unstable laminar boundary layer flow exists over a considerable portion of the

forward half of the airfoil for $Re_c > 7.5 \times 10^5$. However, the boundary layer is completely stabilized at about the 70 per cent chord station.

b) For the ten per cent thick airfoil at $\alpha = 0^\circ$ the unstable region is small until $Re_c > 3 \times 10^6$ and the boundary layer is completely stabilized at about mid-chord.

c) For the six per cent thick airfoil at $\alpha = 4^\circ$ the unstable region is already large on the upper surface at $Re_c = 7.5 \times 10^5$, but is insignificant on the lower surface until $Re_c > 3 \times 10^6$.

d) At Reynolds numbers somewhat below the respective values given in a), b), c), the laminar boundary layer is completely stable.

At $M_1 = 2.0$ the laminar boundary layer is unstable for $Re_c = 7.5 \times 10^5$ over the entire surface of both the six and ten per cent thick airfoils at $\alpha = 0^\circ$.

Conclusions drawn from laminar stability calculations based on the linear perturbation theory must be applied with great care to predictions of transition. However, it seems safe to state that at low supersonic Mach numbers transition on an insulated, symmetrical circular-arc airfoil is probably delayed as compared with transition on an insulated flat plate, for $7.5 \times 10^5 \leqq Re_c \leqq 5.0 \times 10^6$. At angle of attack one would expect a stronger stabilizing effect on the lower surface than on the upper surface. The stabilizing effect of negative pressure gradient on these airfoils is expected to increase with thickness ratio, and this effect may have important consequences for [illegible] drag and th[illegible] selecting supersonic airfoils with optimum aerodynamic characteristics.

LIST OF SYMBOLS

The subscript "1" denotes physical quantities in the undisturbed stream ahead of the body; the subscript " δ " denotes quantities at the "edge" of the boundary layer; the subscript "w" denotes quantities at the surface; the subscript "o" denotes local adiabatic stagnation values of the physical quantities.

x	distance along the surface (measured from mid-chord on airfoil)
y	co-ordinate normal to surface
u	component of gas velocity parallel to surface
p	pressure
ρ	density
T	absolute temperature
μ	ordinary coefficient of viscosity
m	exponent in approximate relation $\mu \sim T^m$
γ	ratio of specific heats of gas, c_p/c_v
a	local speed of sound
M	Mach number u/a
δ	boundary layer thickness
δ^*	boundary layer displacement thickness, $\int_0^\infty \left(1-\frac{\rho}{\rho_\delta}\frac{u}{u_\delta}\right)dy$
L	characteristic length (e.g., radius of curvature of circular-arc airfoil)
U_{MAX}	maximum gas velocity in adiabatic, steady flow, $\sqrt{2 c_p T_o}$
U	$\frac{u_\delta}{U_{MAX}}$
t/c	airfoil thickness ratio
s, t	transformed coordinates parallel and normal to surface in Dorodnitzyn method

$\bar{\delta}$ Value of t corresponding to "edge" of boundary layer, where $y = \delta$

τ $t/\bar{\delta}$

ω velocity ratio u/u_δ

λ $\bar{\delta}^2 \dfrac{1}{1-U^2} \dfrac{dU}{ds}$, modified Pohlhausen parameter

Re_o Reynolds number, $\dfrac{\rho_o U_{Max} L}{\mu_o}$

Re_δ, Re_{δ^*} Reynolds numbers, respectively $\dfrac{\rho_\delta u_\delta \delta}{\mu_\delta}$ and $\dfrac{\rho_\delta u_\delta \delta^*}{\mu_\delta}$,

$Re_{\delta_{cr_{min}}}$ Minimum critical Reynolds number, or stability limit, for laminar flow

c airfoil chord

Re_c Reynolds number, $\dfrac{\rho_1 u_1 c}{\mu_1}$

R radius of curvature of circular-arc airfoil $\frac{1}{2}\left[\frac{t}{c} + \left(\frac{t}{c}\right)^{-1}\right]$

ξ x/c

STABILITY OF THE SUPERSONIC BOUNDARY LAYER WITH A PRESSURE GRADIENT

1. Introduction

Both theoretically and experimentally the strong effects of a pressure gradient in the flow direction on the stability of the laminar boundary layer, the rate of amplification of unstable disturbances, and transition to turbulent flow are well-known at low speeds. Extension of stability considerations based on the small perturbation theory to a compressible fluid (references 1 and 2) opens the way for an analysis of the effect of pressure gradient on laminar boundary layer stability at high speeds. First, however, it was necessary to show that at high Reynolds numbers only the local velocity and temperature distributions across the boundary layer determine the stability of the local flow. A proof of this fact, which amounts to an extension of Pretsch's result (reference 3) to a compressible fluid, together with a careful examination of the approximations involved, has been furnished by Mr. Sin-I Cheng*.

On the assumption that only local flow properties determine local stability, the probable effects of pressure gradient on laminar stability were briefly discussed at the end of reference 2. From physical considerations and a study of the equations of motion, it appears (for example) that the stabilizing influence of a negative pressure gradient must compete with the destabilizing influence of aerodynamic heating, or viscous dissipation, at least for zero heat transfer at the surface. At high supersonic Mach numbers the moderate pressure gradients generally encountered on airfoils

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* Paper to appear shortly.

with continuous slope can be expected to exert only a negligible influence on laminar boundary layer stability. On the other hand, at low supersonic Mach numbers, where the aerodynamic heating effect is still moderate, the laminar boundary layer is theoretically completely stabilized if the negative pressure gradient exceeds a certain critical value that depends on the Mach number, the surface heat transfer rate, and the properties of the gas. Thus, at low supersonic Mach numbers, the stability of the laminar boundary layer over an airfoil is expected to depend critically on the shape, thickness, and angle of attack of the airfoil, while at high supersonic Mach numbers the surface heat transfer rate is the important factor.

The purpose of the present report is to supplement purely qualitative considerations with a quantitative estimate of the stabilizing influence of a favorable (negative) pressure gradient on the laminar boundary layer over representative supersonic airfoils. The methods developed are applicable to the laminar boundary layer over any surface (e.g., nozzles, diffusers, etc.), so long as the approximations of the boundary layer theory remain valid. In order to separate the effects of surface heat transfer rate and pressure gradient for the present, only the case of zero heat transfer is considered.

Because of the extreme sensitivity of the stability of the laminar boundary layer flow to the distribution of the gradient of the product of density and vorticity $\frac{d}{dy}\left(\rho\frac{du}{dy}\right)$ across the layer, it would be desirable to obtain exact solutions of the boundary layer equations of motion for the flow with pressure gradient along the surface. While no exact solutions have been obtained up to the present, Stewartson shows in a recent paper (reference 4) that, at least for Prandtl number unity and a linear viscosity-temperature relation, any compressible fluid boundary layer flow over an insulated surface with a prescribed pressure distribution can be reduced to an equivalent low-speed boundary layer flow with a transformed pressure distribution. This equivalent low-speed flow can then be treated by Howarth's series-expansion method (reference 5), or by other means.

Even with the great simplification introduced by Stewartson, the amount of work involved in the calculation is considerable. As a first attempt, therefore, it was decided to employ the less accurate method of Dorodnitzyn (reference 6), which amounts to an extension to a compressible fluid of the Pohlhausen technique based on the von Kármán momentum equation (reference 7). Laminar stability calculations based on mean velocity and temperature distributions obtained by Dorodnitzyn's method can furnish only the order of magnitude of the pressure gradient effect. However, these calculations should indicate the range of Mach numbers where the pressure gradient plays a critical role in determining laminar boundary layer stability. More accurate flow solutions based on the Stewartson-Howarth method can be obtained later in these critical cases. The approximate laminar stability calculations in the present report might also serve as a guide to experimental research on the effect of pressure gradients on transition on supersonic airfoils, and may have interesting implications for supersonic airfoil drag and airfoil design*.

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* After the calculations of the present report had been completed, it was learned that H. Weil of General Electric, in a restricted report, had carried out similar calculations utilizing Dorodnitzyn's method, with a sixth degree polynomial for the velocity distribution. Because of a numerical error involving a factor of π in an important stability function, the G. E. report came to entirely different (and erroneous) conclusions concerning the effect of pressure gradient on laminar stability at low supersonic Mach numbers. This error was pointed out to Dr. Weil, and it is understood that a corrected and condensed version of his report is to appear shortly in the Journal of the Aeronautical Sciences.

2. Mean Velocity and Temperature Distributions Across the Laminar Boundary with a Pressure Gradient

The development of the boundary layer, and the mean velocity and temperature distributions across the layer, are to be calculated by the Pohlhausen method as modified by Dorodnitzyn for compressible flow over an insulated plane surface with Prandtl number equal to 1.0 (reference 6). The starting point of the Pohlhausen method is the von Kármán integral equation for the momentum balance in the boundary layer (reference 7):

$$\text{(1)}\qquad \frac{\partial}{\partial x}\int_0^\delta \rho u^2\,dy - u_\delta \frac{\partial}{\partial x}\int_0^\delta \rho u\,dy = \delta \rho_\delta u_\delta \frac{du_\delta}{dx} - \mu_w\left(\frac{\partial u}{\partial y}\right)_w$$

Dorodnitzyn noticed that this equation is reduced to a form similar to that for isothermal low-speed flow if the coordinates parallel and normal to the surface are modified as follows:

$$\text{(2a)}\qquad ds = \frac{1}{L\,Re_0}\;\frac{p_\delta}{p_0}\,dx$$

$$\text{(2b)}\qquad dt = \frac{1}{L}\,\frac{\rho}{\rho_0}\,dy$$

With the introduction of these new variables, equation (1) becomes:

$$\text{(3)}\qquad \bar{\delta}\frac{d}{ds}\left\{\upsilon^2\bar{\delta}\int_0^1 w^2\,d\tau\right\} - \upsilon\,\bar{\delta}\frac{d}{ds}\left\{\upsilon\,\bar{\delta}\int_0^1 w\,d\tau\right\}$$

$$= \upsilon f(s)\,\bar{\delta}^2\int_0^1 (1-\upsilon^2 w^2)\,d\tau - \upsilon\left(\frac{dw}{d\tau}\right)_{\tau=0}$$

where $$\delta = L\bar{\delta}\left(1+\frac{\gamma-1}{2}M_r^2\right)^{\gamma/\gamma-1}\int_0^1 (1-U^2w^2)\,d\tau ,$$

$$f(s) = \frac{1}{1-U^2}\frac{dU}{ds}$$

The temperature velocity relation, (3a) $\frac{T}{T_\delta} = 1+\frac{\gamma-1}{2}M_\delta^2(1-w^2)$ given by Crocco (reference 8) for Prandtl number one and zero heat transfer at the surface is employed throughout.

From the boundary condition for $\frac{\partial^2 u}{\partial y^2}$ at the surface, one finds that $\left(\frac{d^2w}{d\tau^2}\right)_{\tau=0} = -\bar{\delta}^2 f(s)$. If (following Pohlhausen) the quantity $\bar{\delta}^2 f(s)$ is now selected as the pressure gradient parameter $\lambda(s)$, then the velocity distribution can be approximated as a polynomial in τ across the boundary layer, with coefficients dependent only on λ, exactly as in the case of incompressible flow. With the fourth degree polynomial of the form:

$$(4) \qquad w = A\tau + B\tau^2 + C\tau^3 + D\tau^4 ,$$

the boundary conditions $$w(0) = 0 , \quad \left(\frac{d^2w}{d\tau^2}\right)_{\tau=0} = \lambda$$
$$w(1) = 1 , \quad \left(\frac{dw}{d\tau}\right)_{\tau=1} = 0$$

are satisfied if $$A = 2+\frac{\lambda}{6} , \quad B = -\lambda/2$$
$$C = \lambda/2 - 2 , \quad D = 1-\frac{\lambda}{6}$$

By substituting the polynomial approximation for the velocity profile (equation 4) into equation (3), a first-order differential equation for λ (s) is obtained. Actually, the non-dimensional variable "s" disappears and the differential equation for $\lambda(x)$ can be written as follows:

$$(5) \qquad \frac{d\lambda}{dx} = S_1(\lambda)\,N_1(x) + S_2(\lambda)\,N_2(x) ,$$

where

$$* \quad (5a) \qquad S_1(\lambda) = \frac{213.12 - 1.92\lambda - 0.20\lambda^2}{213.12 - 5.76\lambda - \lambda^2}$$

$$(5b) \qquad S_2(\lambda) = \frac{7257.6 - 1336.32\lambda + 37.92\lambda^2 + 0.80\lambda^3}{213.12 - 5.76\lambda - \lambda^2}$$

and

$$** \quad (5c) \qquad N_1(x) = \frac{p_\delta''}{p_\delta'} - \frac{\left(\frac{\gamma-3}{2}M_0^2 - 1\right)}{\gamma M_0^2}\,\frac{p_\delta'}{p_\delta}$$

$$(5d) \qquad N_2(x) = -\frac{1}{\gamma M_0^2}\,\frac{p_\delta'}{p_\delta} \quad , \text{ where } M_0^2 = \frac{M_\delta^2}{1+\frac{\gamma-1}{2}M_\delta^2}$$

(The primes denote differentiation with respect to x.)

When the pressure distribution $p_1(x)$ along the surface is prescribed and the value of λ at one point is known, then equation (5) can be integrated by suitable numerical methods. For example, for a supersonic

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* These functions are identical with Pohlhausen's.

** A typographical error in the expression for $N_1(x)$ in Princeton Aero. Eng. Lab. Report #143 (1949) is here corrected. As $M_0 \to 0$, or $M_\delta \to 0$, $N_1(x) \to \frac{U''}{U'}$ and $N_2(x) \to \frac{U'}{U}$, which agrees with Pohlhausen's formulation for $M \ll 1$.

airfoil with attached leading-edge shock, the boundary condition is $\lambda = 0$ at $x = 0$. The general properties of equation (5) for the case of compressible flow are quite similar to the properties in the limiting case of "incompressible" flow.* If a pressure minimum occurs at some point along the surface $(p_1' = 0)$, it is more convenient to introduce the new variable $Z = \frac{\lambda}{p_\delta'}$, in order to remove the singularity in equation (5). Because of the form of $S_1(\lambda)$ and $S_2(\lambda)$ [equations (5a) and (5b)], which stems from the polynomial approximation to the velocity profile (equation 4), the Pohlhausen method cannot be extended beyond $\lambda = +12$ without modification; of course, the value $\lambda = -12$ denotes flow separation. In the case of the smooth supersonic circular-arc airfoils treated in the present report, none of these difficulties arise in the regions of interest for the laminar stability calculations.

Once the distribution $\lambda(x)$ along the surface is known, the velocity distribution $w(\tau)$ across the boundary layer at any station is obtained from equation (4), and the temperature distribution is calculated from the Crocco relation (3a). The actual distance from the surface y/δ is given in terms of τ and λ by the following expression (see equation 2b):

(6) $$y/\delta = \frac{I(\tau, \lambda)}{I_1(\lambda)},$$

where,

(6a) $$I(\tau, \lambda) = \left(1 + \frac{\gamma-1}{2} M_\delta^2\right)\tau - \frac{\gamma-1}{2} M_\delta^2 \int_0^\tau w^2 \, d\tau ,$$

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* See, for example, Goldstein, Modern Fluid Dynamics, Vol. 1, pp. 156-163.

$$(6b)\quad \int_0^1 w^2 d\tau = \frac{1}{3}A^2\tau^3 + \frac{AB}{2}\tau^4 + \left(\frac{B^2+2AC}{5}\right)\tau^5 + \left(\frac{AD+BC}{3}\right)\tau^6 + \left(\frac{C^2+2BD}{7}\right)\tau^7 + \frac{CD}{4}\tau^8 + \frac{D^2}{9}\tau^9$$

$$(6c)\quad I_1(\lambda) = \left(1+\frac{\gamma-1}{2}M_\delta^2\right) - \frac{\gamma-1}{2}M_\delta^2\left[0.5825 - 0.0094\lambda + 0.00611\lambda^2\right]$$

(It turns out that all the boundary stability parameters can be expressed in terms of τ).

The development of the boundary layer thickness along the surface is given by the relation: (See Appendix)

$$(7)\quad Re_\delta = \frac{\sqrt{-\lambda}}{\sqrt{\frac{L}{p_\delta}\frac{dp_\delta}{dx}}} \frac{M_\delta^{3/2}\sqrt{\gamma}}{\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^n} \sqrt{\frac{\gamma-1}{2}Re_0}\; I_1(\lambda),$$

with $n = \frac{5\gamma-3}{4(\gamma-1)} - m$, where "m" is the exponent in approximate viscosity-temperature relation $\mu \sim T^m$. For air, $\gamma = 1.4$, $\frac{\gamma-1}{2} = 0.20$, $n = 2.50 - m$.

3. <u>Calculation of the Stability Limit, or Minimum Critical Reynolds Number, $Re_{\delta_{cr_{min}}}$, for the Laminar Boundary Layer with a Pressure Gradient. Application to Insulated Supersonic Circular-Arc Airfoil</u>

In reference 2 an approximate estimate of the stability limit for a boundary layer flow is obtained by observing that the stability limit occurs very nearly when the phase velocity, c, of a neutral subsonic disturbance has its maximum possible value. On this basis, it was found that [equation (5.6), reference 2]:

$$(8)\qquad Re_{\delta_{cr_{min}}} \approx \frac{25\left[\frac{T}{T_\delta}(c_0)\right]^{1.76}\left(\frac{\partial w}{\partial y/\delta}\right)_{y=0}}{c_0^{\,4}\sqrt{1-M_\delta^2(1-c_0)^2}},$$

where c_0 is the value of c for which the stability function $(1 - 2\bar{\lambda})v$ equals 0.580.* The functions v and $\bar{\lambda}$ are defined as follows:

$$(9a)\qquad v(c) = \frac{-\pi\left(\frac{\partial w}{\partial y/\delta}\right)_{y=0} c}{T_w/T_\delta}\left[\frac{(T/T_\delta)^2}{\left(\frac{\partial w}{\partial y/\delta}\right)^3}\frac{\partial}{\partial y/\delta}\left(\frac{1}{T/T_\delta}\frac{\partial w}{\partial y/\delta}\right)\right]_{w=c}$$

$$(9b)\qquad \bar{\lambda}(c) = \frac{y_c/\delta\left(\frac{\partial w}{\partial y/\delta}\right)_{y=0}}{c} - 1$$

For a neutral subsonic disturbance to exist, it is necessary that $c > 1 - \frac{1}{M_\delta}$.

When the mean velocity and temperature distributions across the boundary layer are calculated by Dorodnitzyn's method for an insulated surface with Prandtl number unity, then all quantities required in the laminar stability calculations can be expressed in terms of λ, M_δ and $w(\tau)$ and its derivatives, as follows: (See Appendix)

$$(10)\qquad Re_{\delta_{cr_{min}}} \approx \frac{25\left[\frac{T}{T_\delta}(c_0)\right]^{1.76}}{c_0^{\,4}\sqrt{1-M_\delta^2(1-c_0)^2}}\cdot\frac{\left(2+\frac{\lambda}{6}\right)I_1(\lambda)}{1+\frac{\gamma-1}{2}M_\delta^2}$$

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* The function λ (c) appearing in reference 2 is here denoted $\bar{\lambda}$ (c), in order to avoid confusion with the Pohlhausen parameter λ (x).

where $\frac{T}{T_\delta}(\varsigma_0) = 1 + \frac{\gamma-1}{2} M_\delta^2 (1-\varsigma_0^2)$, and $I_1(\lambda)$ is given by equation (6c).

Also,

(11a) $$v(c) = \frac{-\pi\left(2+\frac{\lambda}{6}\right) c}{\left(\frac{dw}{d\tau}\right)^2_{w=c}} \left(\frac{1+\frac{\gamma-1}{2} M_\delta^2 (1-c^2)}{1+\frac{\gamma-1}{2} M_\delta^2}\right)^2 \left[\frac{\frac{d^2 w}{d\tau^2}}{\frac{dw}{d\tau}} + \frac{2(\gamma-1) M_\delta^2 w \frac{dw}{d\tau}}{1+\frac{\gamma-1}{2} M_\delta^2 (1-w^2)}\right]_{w=c}$$

(11b) $$\bar{\lambda}(c) = \frac{2+\frac{\lambda}{6}}{c} \cdot \frac{I(\lambda, \tau_c)}{1+\frac{\gamma-1}{2} M_\delta^2}$$, where $I(\lambda, \tau_c)$ is given

by equations (6a) and (6b), and $w(\tau)$ and its derivatives are obtained from equation (4).

Once the pressure distribution is known over a surface, the methods just outlined can be applied to the calculation of the variation of Re_δ and $Re_{\delta_{cr_{min}}}$ with distance along the surface. When $Re_\delta < Re_{\delta_{cr_{min}}}$ the local boundary layer flow is stable and all small disturbances are damped out. When $Re_\delta > Re_{\delta_{cr_{min}}}$ the local laminar boundary layer flow is unstable and self-excited disturbances of definite wave-lengths appear in the flow. If the unstable region is sufficiently extensive, these disturbances eventually grow large enough as they move downstream to destroy the laminar motion and cause transition to turbulence.

One physically interesting case where the effect of pressure gradient on laminar stability might have important implications is the flow over supersonic airfoils. As an illustrative example, the stability limits were calculated for symmetrical circular-arc airfoils of six and ten per cent thickness ratio at $M_1 = 1.5$, 2.0, and 3.0, and at angles of attack of zero and four degrees. Pressure distributions over these airfoils were calculated by standard methods, utilizing simple oblique shock theory for the flow across the leading-edge waves, and Prandtl-Meyer expansion for the flow over the surface behind the leading edge (references 9 and 10). This method neglects the entropy gradients in the flow generated by the reflection of the expansion waves from the leading-edge shocks, but these effects are very small in the cases considered here.

For the symmetric circular-arc airfoil the relations utilized in the Dorodnitzyn method take a particularly simple form. The differential equation (5) for λ is:

(12) $$\frac{d\lambda}{d\xi} = \frac{c}{2} N_1(\xi) S_1(\lambda) + \frac{c}{2} N_2(\xi) S_2(\lambda) \quad * \qquad \text{with } \lambda(-1.00) = 0,$$

where, (12a), $$\frac{c}{2} N_1(\xi) = -\frac{1}{\left(\frac{R}{c/2}\right)} \left[\frac{1 + 2.20 M_\delta^2 - 2 M_\delta^4}{(M_\delta^2 - 1)^{3/2}}\right]$$

(12b) $$\frac{c}{2} N_2(\xi) = \frac{1}{\left(\frac{R}{c/2}\right)} \left[\frac{1 + 0.20 M_\delta^2}{\sqrt{M_\delta^2 - 1}}\right] \quad ; \text{air, } \gamma = 1.4,\ m = 0.76$$

(13) $$Re_\delta = \frac{\sqrt{\lambda}\,(M_\delta^2 - 1)^{1/4} M_\delta}{(1 + 0.20 M_\delta^2)^{1.74}} \quad \frac{Re_{0,R}}{\sqrt{5}} \quad I_1(\lambda) \,,$$

$$Re_c = \frac{M_1}{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{2.24}} \quad \frac{Re_{0,R}}{\sqrt{5}} \quad \frac{1}{(R/c)} \quad \frac{\bar{p}_0}{p_0}$$

(Note that the characteristic length L is taken to be R) where p_o is the

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* Here c is used as the symbol for chord length.

stagnation pressure behind the leading edge shock and $\bar{p}_0$ is stagnation pressure in the undisturbed stream.

The differential equation (12) was integrated by a finite difference method in which steps of 0.10c were utilized. ($\Delta\xi = 0.20$). Several checks were made with $\Delta\xi = 0.10$ and the largest error found in the values of λ at any station was less than 2%. A typical variation of $\lambda(\xi)$ over an airfoil surface is illustrated in Table 1 for t/c = 0.06, $\alpha = 0^\circ$, $M_1 = 1.5$, 2.0 and 3.0. A typical calculation of $Re_{\delta_{cr_{min}}}$ is recorded in Table 2, for $\xi = -0.60$, $t/c = 0.06$, $\alpha = 0^\circ$, $M_1 = 1.5$.

4. <u>Discussion of Results of Stability Calculations for Insulated Symmetrical Circular-Arc Airfoils at $M_1 = 1.5$, 2.0, and 3.0</u>.

In figures 1a and 1b, the chordwise distribution of $Re_{\delta_{cr_{min}}}$ for insulated, symmetrical circular-arc airfoils of six and ten per cent thickness ratio at $\alpha = 0^\circ$ and 4° is plotted for $M_1 = 1.5$, 2.0, and 3.0. Several important conclusions can be drawn from the figures.

1) Near the leading-edge, where the boundary layer is still relatively thin and the viscous shear stress is large, the effect of pressure gradient on the velocity and temperature distributions across the boundary layer is small. This fact is expressed directly in the Pohlhausen parameter, $\lambda \sim \frac{\delta^2}{\nu}\frac{du_\delta}{dx}$. Consequently, near the leading edge the effect of aerodynamic heating is dominant, and the values of $Re_{\delta_{cr_{min}}}$ are not much larger than the stability limit for an insulated flat plate at comparable Mach numbers.

2) Toward mid-chord, where the boundary layer is thicker and λ is higher, the stabilizing effect of a negative pressure gradient is significant

at $M_1 = 1.50$. At $\alpha = 0°$, the laminar boundary layer is completely stabilized at about mid-chord for $t/c = 0.10$, ($\lambda \doteq 6$) and at about 70 per cent chord for $t/c = 0.05$ ($\lambda \doteq 6.5$). The stronger pressure gradients on the thicker airfoil are largely responsible for earlier stabilization.

3) At $M_1 = 2.0$ the destabilizing effect of aerodynamic heating is the dominant factor over the entire airfoil and the increase in $Re_{\delta_{cr_{min}}}$ with λ is moderate. At $M_1 = 3$, the influence of pressure gradient on laminar stability is negligible, at least for an insulated surface.

4) At $M_1 = 1.50$ and $M_1 = 2.0$ the main effect of angle of attack is to produce higher values of $Re_{\delta_{cr_{min}}}$ on the lower surface than on the upper surface. The lower Mach number and also the stronger pressure gradients on the lower surface are largely responsible for this effect.

All the observations (1) - (4) can be anticipated from a study of the general relation $Re_{\delta^*_{cr_{min}}} = g(\lambda, M_\delta)$, which is plotted in figure 5 for $M_\delta = 1.5$, 1.75 and 2.0.* For comparison, the curve $Re_{\delta^*_{cr_{min}}} = g_o(\lambda)$ obtained by Pretsch (reference 3) for $M_1 << 1$ is also shown. For low pressure gradients the stabilizing effect of negative pressure gradient is naturally much stronger for $M_1 << 1$ than for $M_1 > 1$. However, $Re_{\delta^*_{cr_{min}}} \longrightarrow 14 \times 10^3$ (finite) as $\lambda \rightarrow +12$ at low speeds, while complete stabilization of the laminar boundary layer over an insulated surface is achieved at low supersonic Mach numbers above a certain critical value of λ that depends only upon the Mach number and the properties of the gas. The effect of a pressure gradient along the surface on the stability of the

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* $Re_{\delta^*_{cr_{min}}}$ is given by equation 10 with $I_1(\lambda)$ replaced by $I_2(\lambda)$, where $I_2(\lambda) = \int_0^1 \left\{1 + \frac{\gamma-1}{2} M_\delta^2 (1-w^2) - w\right\} d\tau = b_0 + b_1\lambda + b_2\lambda^2$, and $b_0 = 0.30 + 0.4175 \frac{\gamma-1}{2} M_\delta^2$, $b_1 = -\left[0.0083 + 0.0094 \frac{\gamma-1}{2} M_\delta^2\right]$
$b_2 = -0.0001 \frac{\gamma-1}{2} M_\delta^2$

laminar boundary layer, although similar in some respects to the effect of surface heat transfer rate, differs essentially in its dependence on the boundary layer thickness, as expressed, for example, by the Pohlhausen parameter. At a given Mach number the stability limit on a flat plate with zero pressure gradient depends only on the ratio of the "effective" temperature difference $T_s - T_w$, to the free stream temperature, where T_s is the surface temperature at zero heat transfer, and T_w is the existing surface temperature (reference 2).

Since the distribution of the gradient of the product of density and vorticity $\frac{d}{dy}\left(\rho \frac{du}{dy}\right)$ across the boundary layer largely determines the laminar stability limit, a clearer insight into the effect of pressure gradient on laminar stability is obtained by calculating the velocity and temperature profiles. (Figures 2 and 3, $t/c = 0.06$, $\alpha = 0^\circ$, $\xi = -0.60$. (20% chord station)). For $M_1 = 1.5$ the convexity of the velocity profile is apparently sufficient to insure that the effect of negative pressure gradient is stabilizing, despite the rate of increase of gas density outward from the surface. At higher Mach numbers, however, the rate of increase of density outward from the surface is so large for $w > 1 - \frac{1}{M_1}$ that the quantity $\frac{d}{dy}\left(\rho \frac{du}{dy}\right)$ certainly attains smaller negative values at a given w and may even become positive. This behavior is illustrated by the variation of the stability function $(1 - 2\overline{\lambda})v$ across the boundary layer (sub-plot in figure 3).

It is clear that the effect of heat transfer on laminar boundary layer stability at high Mach numbers is fundamentally different than the effect of pressure gradient. Heat withdrawal from the gas to the surface stabilizes the flow largely because it produces an initial rate of <u>decrease</u> of gas density outward from the surface. The effect of a favorable pressure gradient, on the other hand, can appear only in the relative convexity of the velocity profile, and at high Mach numbers cannot counterbalance the destabilizing influence of the density increase outward from the surface. As the Mach number is increased, the velocity profile also appears to be less strongly convex with given pressure gradient. As shown in reference 2, for an insulated surface,

$$\left[\frac{d}{d y/\delta}\left(\frac{\rho}{\rho_\delta}\frac{dw}{dy/\delta}\right)\right]_{y=0} = -\frac{1}{\left(\frac{T_w}{T_\delta}\right)^{m+1}}\frac{\delta^2}{\nu}\frac{du_\delta}{dx},$$

but

$$\left[\frac{d^2}{d(y/\delta)^2}\left(\frac{\rho}{\rho_\delta}\frac{dw}{dy/\delta}\right)\right]_{y=0} = \sigma(m+1)(\gamma-1)M_\delta^2\frac{\left(\frac{dw}{dy/\delta}\right)^3_{y=0}}{\left(\frac{T_w}{T_\delta}\right)^2} > 0$$

$\sigma = \frac{c_p \mu}{k}$, Prandtl number

*

So far as the stabilizing effect of negative pressure gradient is concerned, the interesting situations occur at low supersonic Mach numbers. Accordingly, the regions of laminar boundary layer stability and instability

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* Of course the Dorodnitzyn-Pohlhausen method of approximating the velocity profile by a fourth degree polynomial (equation 4) does not permit the proper boundary condition $\left(\frac{d^3w}{d\tau^3}\right)_{\tau=0} = 2(m-1)\left(\frac{dw}{d\tau}\right)^3_{\tau=0} < 0$ to be satisfied. In fact, with $w(\tau)$ given by equation 4,

$$\left(\frac{d^3w}{d\tau^3}\right)_{\tau=0} = 3\lambda - 12$$

If $m = 0.76$ (air, room temperature), then for $\lambda > 2$(approx), the value of on the symmetrical circular-arc airfoils at $M_1 = 1.50$ are illustrated in figures 4a - 4d; for comparison, the case $t/c = 0.06$, $\alpha = 0^\circ$, $M_1 = 2.0$ is also included (figure 4e). The growth of Re_δ along the surface is plotted for flight Reynolds numbers of 7.5×10^5, 3×10^6, 15×10^6 and 30×10^6, based on airfoil chord and physical quantities evaluated in the undisturbed stream.

For a sufficiently large flight Reynolds number, the stability limit ($Re_\delta = Re_{\delta_{cr_{min}}}$) is crossed just aft of the leading edge in every case. Downstream of this point self-excited laminar disturbances appear in the boundary layer flow and grow steadily as they propagate along the surface, up to the station at which a stability limit, or neutral point, is again reached. For $t/c = 0.06$, $\alpha = 0^\circ$, an unstable region exists over a considerable portion of the airfoil surface when $Re_c > 7.5 \times 10^5$, while for $t/c = 0.10$, the unstable region is small until $Re_c > 3 \times 10^6$. At $\alpha = 4^\circ$, ($t/c = 0.06$) the unstable region on the upper airfoil surface is already large at $Re_c = 7.5 \times 10^5$, while on the lower surface the unstable region is insignificant until $Re_c > 3 \times 10^6$.* At flight Reynolds numbers somewhat below these values the laminar boundary layer over the airfoil is completely stable, at least on the basis of the inviscid flow pressure distribution. Two important questions immediately arise:

Footnote continued from p. 15.

$\left(\frac{d^3\omega}{d\tau^3}\right)_{\tau=0}$ is smaller negatively than it should properly be, and is even positive for $\lambda > 4$. This fact by itself would mean that the values of $Re_{\delta_{cr_{min}}}$ calculated in the present report are probably somewhat too low for $\lambda > 2$ (approx.), and that complete stabilization of the laminar boundary layer probably occurs at lower values of λ at low supersonic Mach numbers than indicated here. However, it is difficult to estimate the effect of errors in the higher derivatives and elsewhere in the Dorodnitzyn method. More accurate calculations of the velocity profile are obviously required.

* It is interesting to compare this behavior with the low-speed case (reference 3).

1) Do the self-excited laminar boundary layer disturbances have sufficient time to grow as they propagate through the unstable region along the airfoil surface so that transition to turbulent flow occurs before the stable region is reached?

2) Suppose that transition to turbulent flow has already occurred before the point is reached at which a laminar flow would theoretically be completely stabilized: Is it possible for the turbulent flow to revert back to laminar flow downstream of the point at which $Re_\delta \simeq Re_{\delta_{cr_{min}}}$, or is transition an essentially irreversible process?

A linear perturbation theory can never pretend to answer such questions: at most it can furnish only the initial rate of amplification of the unstable disturbances. The solution to the transition problem depends on a knowledge of the turbulent energy spectrum, the rate of amplification of the unstable disturbances, and the process, treated only qualitatively as yet, by which the laminar flow is destroyed and transition occurs. All that can be safely stated here is that at low supersonic Mach numbers transition on a symmetrical circular-arc airfoil is probably delayed as compared with transition on a flat plate, at least when $7.5 \times 10^5 < Re_c < 5.0 \times 10^6$ (say). At angle of attack, one would expect a larger stabilizing effect on the lower surface than on the upper surface. The stabilizing effect on the laminar boundary layer flow should increase with airfoil thickness ratio, and this effect may have important consequences for skin-friction drag, and for the problem of selecting airfoils with optimum characteristics.

5. Some Problems for Future Investigation

1. Calculations of the stability limits for supersonic laminar boundary layer flows in representative cases with the aid of the Dorodnitzyn

method have made it clear that accurate solutions of the boundary layer equations with pressure gradient are required at low supersonic Mach numbers ($1 \leqq M \leqq 2$). The Stewartson-Howarth method could be employed in a few cases of particular interest, and the distributions of $\frac{d}{dy}\left(\rho \frac{du}{dy}\right)$ across the boundary layer compared with those obtained by the Dorodnitzyn method.

2. Since Kalikhman (reference 11) has extended the Dorodnitzyn method to include surface heat transfer (at least for Prandtl number unity), it would be desirable to utilize Kalikhman's method to calculate the effect of pressure gradient on the stabilizing influence of heat withdrawal at subsonic and supersonic Mach numbers. It would be particularly interesting to determine the effect of pressure gradient on the critical heat transfer rate required for complete stabilization of the laminar boundary layer.

3. The conclusion that the laminar boundary layer at low supersonic Mach numbers is completely stabilized when the pressure gradient parameter λ is larger than a certain critical value could be tested by an experimental investigation of the effect of pressure gradient on transition.

6. Conclusions

1. At low supersonic Mach numbers the laminar boundary layer over an insulated surface is completely stabilized by a negative pressure gradient larger than a certain critical value that depends only on the Mach number and the properties of the gas. When the Dorodnitzyn-Pohlhausen method is employed to calculate the boundary layer development, and the velocity distribution across the layer is approximated by a fourth-degree polynomial, complete laminar stabilization is found at $\lambda \cong 6.5$ for $M_1 = 1.5$, at

$\lambda \cong 7.0$ for $M_1 = 1.75$, and $\lambda \cong 11$ for $M_1 = 2.0$ (for example). (Here $\lambda \sim \frac{\delta^2}{\nu} \frac{du_\delta}{dx} f(M)$ is the modified Pohlhausen parameter). For $M_1 = 2.0$ the destabilizing effect of aerodynamic heating is dominant and the increase of the minimum critical Reynolds number, $Re_{\delta_{cr_{min}}}$, or stability limit, with λ is small until $\lambda \to 11$. For $M_1 = 3.0$ the influence of negative pressure gradient is negligible, at least for an insulated surface.

2. By examining the distribution across the boundary layer of the gradient of the product of density and vorticity, $\frac{d}{dy}\left(\rho \frac{du}{dy}\right)$, which largely determines the limit of the stability of the flow, it is clear that the stabilizing effect of a negative pressure gradient at high Mach numbers operates in a fundamentally different way than the stabilizing effect of heat withdrawal from gas to surface. At high Mach numbers heat withdrawal stabilizes the flow largely because of the initial rate of <u>decrease</u> of gas density outward from the surface. On the other hand, a negative pressure gradient over an insulated surface can affect only the relative convexity of the velocity profile, and cannot counterbalance the destabilizing influence of the <u>increase</u> of density outward from the surface.

3. Calculation of the chordwise distribution of $Re_{\delta_{cr_{min}}}$ for insulated, symmetrical, supersonic circular-arc airfoils of six and ten per cent thickness ratio at $\alpha = 0^\circ$ and 4° for $M_1 = 1.5$ show that:

a) Near the leading edge, where the boundary layer is thin, and λ is small, $Re_{\delta_{cr_{min}}}$ is not much larger than the value for an insulated flat plate at comparable Mach numbers;

b) At $M_1 = 1.5$, the stabilizing effect of negative pressure gradient becomes significant toward mid-chord. At $\alpha = 0^\circ$ the laminar boundary layer is completely stabilized at about 50 per cent chord on the 10 per cent thick airfoil, and at about 70 per cent chord for $t/c = 0.06$.

c) The main effect of angle of attack ($\alpha = 4^{o}$) at $M_1 = 1.5$ is to produce higher values of $Re_{\delta_{cr_{min}}}$ on the lower surface than on the upper surface.

4. Comparison between the calculated values of the stability limits and the growth of Re_{δ} along the airfoil surfaces for flight Reynolds numbers Re_c of 7.5×10^5, 3×10^6, 15×10^6 and 30×10^6 at $M_1 = 1.5$, show that:

a) For $t/c = 0.06$, $\alpha = 0^{o}$, a region of unstable laminar boundary layer flow exists over a considerable portion of the leading half of the airfoil for $Re_c > 7.5 \times 10^5$;

b) For $t/c = 0.10$, the unstable region is small until $Re_c > 3 \times 10^6$.

c) At $\alpha = 4^{o}$ ($t/c = 0.06$) the unstable region on the upper airfoil surface is already large at $Re_c = 7.5 \times 10^5$, while on the lower surface the unstable region is insignificant until $Re_c > 3 \times 10^6$.

d) For Reynolds numbers below the respective values in a), b), c) the laminar boundary layer is theoretically completely stable.

At $M_1 = 2.0$, the laminar boundary layer is unstable for $Re_c = 7.5 \times 10^5$ over the entire surface of both the six and ten per cent thick airfoils at $\alpha = 0^{o}$.

5. Conclusions drawn from laminar stability calculations based on the linear perturbation theory must be applied with great care to predictions of transition. However, it seems safe to state that at low supersonic Mach numbers and for $7.5 \times 10^5 \leqq Re_c \leqq 5.0 \times 10^6$ transition on an insulated

symmetrical, circular-arc airfoil is probably delayed as compared with transition on an insulated flat plate. At angle of attack one would expect a stronger stabilizing effect on the lower surface than on the upper surface. The stabilizing effect of negative pressure gradient on these airfoils is expected to increase with the thickness ratio, and this effect may have important consequences for airfoil skin friction drag and on the problem of selecting airfoils with optimum aerodynamic characteristics.

6. On the basis of the estimates of the stabilizing effect of negative pressure gradient obtained in the present report, it seems worthwhile to obtain more accurate solutions of the boundary layer equations at low supersonic Mach numbers by the Stewartson-Howarth method (references 4 and 5), or other means. It would also be interesting to obtain some estimates of the effect of negative pressure gradient on the critical rate of heat withdrawal required for complete stabilization of the laminar boundary layer flow at supersonic speeds. Kalikhman's extension of Dorodnitzyn's method offers a scheme for carrying out the necessary calculations.

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APPENDIX A

1. The Stability Functions $\bar{\lambda}$ (c) and ν (c) in Terms of $w(\tau)$ and Its Derivatives.

Since the stability functions $\bar{\lambda}$ (c) and $\nu(c)$ are non-dimensional, the velocity and temperature derivatives appearing in these functions (equations 9a and 9b) can be expressed in terms of any convenient length parameter. For example:

$$(14a)\qquad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial t}\frac{dt}{dy} = \frac{dw}{d\tau}\,\frac{1}{\delta}\,\frac{dt}{dy},$$

or,

$$(14b)\qquad \frac{\partial w}{\partial y/\delta} = \frac{dw}{d\tau}\frac{dt}{dy} = \frac{dw}{d\tau}\,\frac{1}{L}\,\frac{\rho}{\rho_0}$$

It is most convenient to express the velocity and temperature derivatives contained in $\bar{\lambda}$ and ν in terms of y/δ .

Now, $$(15)\qquad \frac{\rho}{\rho_0} = \frac{\rho}{\rho_\delta}\frac{\rho_\delta}{\rho_0} = \frac{\rho_\delta}{\rho_0}\Big/\frac{T}{T_\delta}$$

Therefore,

$$(16)\qquad \frac{\partial w}{\partial y/\delta} = \frac{1}{\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{1/\gamma-1}}\cdot\frac{1}{1+\frac{\gamma-1}{2}M_\delta^2\left(1-w^2\right)}\cdot\frac{1}{L}\frac{dw}{d\tau},$$

for Prandtl number unity and zero heat transfer at the surface. In particular,

$$(16a)\qquad \left(\frac{\partial w}{\partial y/\delta}\right)_{y=0} = \frac{1}{\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{1/\gamma-1}}\;\frac{1}{L}\left(\frac{dw}{d\tau}\right)_{\tau=0}.$$

By differentiating equation (16) one obtains:

(17) $$\frac{\partial^2 w}{\partial (y/\delta)^2} = \left[\frac{1}{\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{1/\gamma-1}}\ \frac{1}{L}\right] \cdot \frac{d}{d\tau}\left\{\frac{1}{1+\frac{\gamma-1}{2}M_\delta^2(1-w^2)}\ \frac{dw}{d\tau}\right\} \cdot \frac{dt}{dy},$$

or

(18) $$\frac{\dfrac{\partial^2 w}{\partial (y/\delta)^2}}{\left(\dfrac{\partial w}{\partial y/\delta}\right)} = \frac{1/L}{\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{1/\gamma-1}} \cdot \frac{1}{\left[1+\frac{\gamma-1}{2}M_\delta^2(1-w^2)\right]}$$

$$\times \left[\frac{\dfrac{d^2w}{d\tau^2}}{\dfrac{dw}{d\tau}} + \frac{(\gamma-1)M_\delta^2\, w\, \dfrac{dw}{d\tau}}{1+\frac{\gamma-1}{2}M_\delta^2(1-w^2)}\right]$$

For Prandtl number unity,

(19) $$\frac{\partial T/T_\delta}{\partial y/\delta} = -(\gamma-1)M_\delta^2\, w\, \frac{\partial w}{\partial y/\delta},$$

by differentiation of the Crocco relation (equation 3a). Therefore,

(20) $$\frac{1}{T/T_\delta}\ \frac{\partial T/T_\delta}{\partial y/\delta} = \frac{1}{L\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{1/\gamma-1}} \cdot \frac{-(\gamma-1)M_\delta^2\, w\, \frac{dw}{d\tau}}{\left[1+\frac{\gamma-1}{2}M_\delta^2(1-w^2)\right]^2}$$

Finally,

(21) $$v(c) = \frac{-\pi\left(\dfrac{\partial w}{\partial y/\delta}\right)_{y=0} c}{T_w/T_\delta}\left[\frac{T/T_\delta}{\left(\dfrac{\partial w}{\partial y/\delta}\right)^2}\left\{\frac{\dfrac{\partial^2 w}{\partial (y/\delta)^2}}{\dfrac{\partial w}{\partial y/\delta}} - \frac{\dfrac{\partial}{\partial y/\delta}(T/T_\delta)}{T/T_\delta}\right\}\right]_{w=c}$$

$$= \frac{-\pi\left(\frac{dw}{d\tau}\right)_{\tau=0} c}{\left(\frac{dw}{d\tau}\right)^2_{w=c}}\left[\frac{1+\frac{\gamma-1}{2}M_\delta^2(1-c^2)}{1+\frac{\gamma-1}{2}M_\delta^2}\right]^2\left[\frac{\dfrac{d^2w}{d\tau^2}}{\dfrac{dw}{d\tau}} + \frac{2(\gamma-1)M_\delta^2\, w\,\dfrac{dw}{d\tau}}{1+\frac{\gamma-1}{2}M_\delta^2(1-w^2)}\right]_{w=c}$$

The function $\bar{\lambda}$ (c) is also expressed in terms of $w(\tau)$ and its derivatives, as follows:

(22) $$\bar{\lambda}(c) = \frac{y_c/_\delta \left(\frac{\partial w}{\partial y/_\delta}\right)_{y=0}}{c} - 1$$

Now, (23)
$$d y/_\delta = L\frac{\rho_0}{\rho}\, d\tau = L\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{1/\gamma-1}\left[1+\frac{\gamma-1}{2}M_\delta^2(1-w^2)\right] d\tau$$

Therefore,

(23a) $$y/_\delta = L\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{1/\gamma-1}\int_0^\tau\left[1+\frac{\gamma-1}{2}M_\delta^2(1-w^2)\right] d\tau \text{ , and}$$

(24) $$\bar{\lambda}(c) = \left\{\frac{\left(\frac{dw}{d\tau}\right)_{\tau=0}}{c}\;\frac{1}{1+\frac{\gamma-1}{2}M_\delta^2}\int_0^\tau\left[1+\frac{\gamma-1}{2}M_\delta^2(1-w^2)\right]d\tau\right\} - 1.$$

(24a) $$\bar{\lambda}(c) = \frac{\left(\frac{dw}{d\tau}\right)_{\tau=0}}{c}\;\frac{I(\tau, \lambda)}{1+\frac{\gamma-1}{2}M_\delta^2} - 1$$

2. Minimum Critical Reynolds Number, $Re_{\delta_{cr_{min}}}$, and Boundary Layer Reynolds Numbers Re_δ and Re_{δ^*}.

In equation (8) the minimum critical Reynolds number based on the boundary layer thickness is expressed in terms of c_0, M_δ and $\left(\frac{\partial w}{\partial y/_\delta}\right)_{y=0}$.
However,

(25) $$\left(\frac{\partial w}{\partial y/\delta}\right)_{y=0} = \frac{\delta}{\bar{\delta}}\left(\frac{\partial w}{\partial y/_{\bar{\delta}}}\right)_{y=0}$$

and the ratio $\frac{\delta}{\bar{\delta}}$ is obtained from the relation (25), as follows:

$$(26)\quad \frac{\delta}{\bar{\delta}} = L\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{1/\gamma-1}\int_0^1\left[1+\frac{\gamma-1}{2}M_\delta^2(1-w^2)\right]d\tau$$

, so that,

$$(27)\quad \left(\frac{\partial w}{\partial y/\delta}\right)_{y=0} = \frac{\left(\frac{dw}{d\tau}\right)_{\tau=0} I_1(\lambda)}{1+\frac{\gamma-1}{2}M_\delta^2}$$

where $I_1(\lambda)$ is the value of the integral in equation (26). The expression for $Re_{\bar{\delta}_{cr_{min}}}$ in equation (10) follows from equation (27).

The relation between $Re_{\bar{\delta}}$ and the Mach number, pressure gradient and λ is derived from the definition of the Pohlhausen parameter, $\lambda = \bar{\delta}^2 f(s)$, where:

$$(28)\quad f(s) = \frac{1}{1-U^2}\frac{dU}{ds} = \left(1+\frac{\gamma-1}{2}M_\delta^2\right)\left(-\frac{U}{\gamma M_\delta^2}\frac{1}{p_\delta}\frac{dp_\delta}{ds}\right) = -\frac{\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{1/2}}{\gamma M_\delta}\sqrt{\frac{\gamma-1}{2}}\frac{1}{p_\delta}\frac{dp_\delta}{ds}$$

Now,

$$(29)\quad \frac{1}{p_\delta}\frac{dp_\delta}{ds} = \frac{L}{p_\delta}\frac{dp_\delta}{dx} Re_0\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{\gamma/\gamma-1},$$

and therefore,

$$(30)\quad f(s) = -\frac{L}{p_\delta}\frac{dp_\delta}{dx}\sqrt{\frac{\gamma-1}{2}}\frac{1}{\gamma M_\delta} Re_0\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{\gamma/\gamma-1+1/2}$$

Consequently,

$$(31)\quad \bar{\delta} = \frac{\sqrt{-\lambda}}{\sqrt{\frac{L}{p_\delta}\frac{dp_\delta}{dx}}}\;\frac{\sqrt{\gamma M_\delta}}{\left(\frac{\gamma-1}{2}\right)^{1/4}\sqrt{Re_0}}\;\frac{1}{\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^{\frac{1}{2}\frac{\gamma}{\gamma-1}+1/4}}$$

The connection between $\bar{\delta}$ and $\bar{\bar{\delta}}$ is furnished by equation (26). By utilizing the relation,

$$(32) \qquad Re_\delta = \frac{\left(\frac{\rho_\delta}{\rho_0}\right)}{\left(\frac{\mu_\delta}{\mu_0}\right)} \frac{a_\delta}{a_0} M_\delta \left(\frac{\gamma-1}{2}\right)^{1/2} Re_0 \, \delta/L \, ,$$

the following expression for Re_δ is finally obtained:

$$(33) \qquad Re_\delta = \frac{\sqrt{-\lambda}}{\sqrt{\frac{L}{\rho_\delta}\frac{d\rho_\delta}{dx}}} \; \frac{M_\delta^{3/2}\sqrt{\gamma}\;\sqrt{\left(\frac{\gamma-1}{2}\right)^{1/2} Re_0}}{\left(1+\frac{\gamma-1}{2}M_\delta^2\right)^n} \; I_1(\lambda)$$

where $n = \dfrac{5\gamma-3}{4(\gamma-1)} - m$

The expression for Re_{δ^*} is obtained as follows: From the definition of the boundary layer displacement thickness,

$$(34) \qquad \frac{\delta^*}{\delta} = \frac{\bar{\bar{\delta}}}{\delta} L \int_0^1 \frac{\rho_0}{\rho_\delta}\frac{\rho_\delta}{\rho}\left(1-\frac{\rho}{\rho_\delta}w\right) d\tau \, ,$$

or,

$$(34a) \qquad \frac{\delta^*}{\delta} = \frac{\int_0^1 \left[1+\frac{\gamma-1}{2}M_\delta^2(1-w^2) - w\right] d\tau}{I_1(\lambda)} = \frac{I_2(\lambda)}{I_1(\lambda)} \, ,$$

and the value of Re_{δ^*} is given by the same expression as Re_δ, with $I_1(\lambda)$ replaced by $I_2(\lambda)$. The formula for I_2 in terms of λ is given in the footnote on page 13.

TABLE 1

Representative Distribution of $\lambda(\xi)$ Over Symmetrical Circular-Arc Airfoils: $\frac{t}{c} = 0.06$, $\alpha = 0°$

ξ	$\lambda(\xi)$		
	$M_1 = 1.5$	$M_1 = 2.0$	$M_1 = 3.0$
-1.00	0.000	0.000	0.000
-0.80	1.251	0.8938	0.8033
-0.60	2.267	1.741	1.628
-0.40	3.153	2.563	2.492
-0.20	3.964	3.380	3.419
0.00	4.737	4.213	4.444
0.20	5.502	5.085	5.623
0.40	6.289	6.034	7.077
0.60	7.137	7.124	9.167
0.80	8.108	8.502	
1.00	9.346	11.103	

TABLE 2

Sample Calculation of Stability Limit, $Re_{\delta_{cr_{min}}}$ $M_1 = 1.5$, $\frac{t}{c} = 0.06$, $\alpha = 0°$, $\xi = -0.60$

$M = 1.353$, $\lambda = 2.2666$, $A = 2.3778$, $B = -1.1333$, $C = -.8667$, $D = .62224$

$$\omega = A\tau + B\tau^2 + C\tau^3 + D\tau^4$$

τ	ω	$\frac{d\omega}{d\tau}$	$\frac{d^2\omega}{d\tau^2}$	$\frac{\frac{d\omega}{d\tau}}{\frac{d^2\omega}{d\tau^2}}$	$.80M^2\omega\frac{d\omega}{d\tau}$	$1+.20M^2(1-\omega^2)$
.215 .220 .225	.4516 .4605 .4696	1.795 1.780 1.764	-3.039 -3.049 -3.059	-1.693 -1.713 -1.733	1.187 1.200 1.213	1.291 1.288 1.285

$\frac{.80M^2\omega\frac{d\omega}{d\tau}}{1+.20M^2(1-\omega^2)}$	$\frac{\frac{d\omega}{d\tau}}{\frac{d^2\omega}{d\tau^2}} + \frac{.80M^2\omega\frac{d\omega}{d\tau}}{1+.20M^2(1-\omega^2)}$	$\pi A\omega$	$-\pi A\omega[1+.20M^2(1-\omega^2)]^2$	$\left[(1+.20M^2)\frac{d\omega}{d\tau}\right]^2$	$\frac{-\pi A\omega[1+.20M^2(1-\omega^2)]^3}{\left[(1+.20M^2)\frac{d\omega}{d\tau}\right]^2}$	$v(\omega)$
.9192 .9315 .9436	-.7741 -.7816 -.7898	3.373 3.440 3.506	-5.625 -5.711 -5.793	6.013 5.912 5.810	-.9356 -.9661 -.9970	.7242 .7551 .7875

τ	$(1+.20M^2)\tau$	$.20M^2\int_0^\tau \omega^2 d\tau$	$(1+.20M^2)\tau - .20M^2\int_0^\tau \omega^2 d\tau$	$\frac{A}{(1+.20M^2)\omega}$	$\bar{\lambda}(\omega)$	$(1-2\bar{\lambda})v$
.215 .220 .225	.2937 .3005 .3074	.005736 .006117 .006513	.2880 .2944 .3009	3.855 3.780 3.708	.1100 .1129 .1157	.5648 .5847 .6052

From Figure 3 (sub-plot), $(1-2\bar{\lambda})v = 0.580$
for $c_o = \omega = .4585$, or $Re_{\delta_{cr_{min}}} = 2590.0$

$$Re_{\delta_{cr_{min}}} = \frac{25\ A\ I_1(\lambda)\ \left[1+.20M^2(1-c^2)\right]^{1.76}}{c_o^4\ (1+.20M^2)\sqrt{1-M^2(1-c_o)^2}}$$

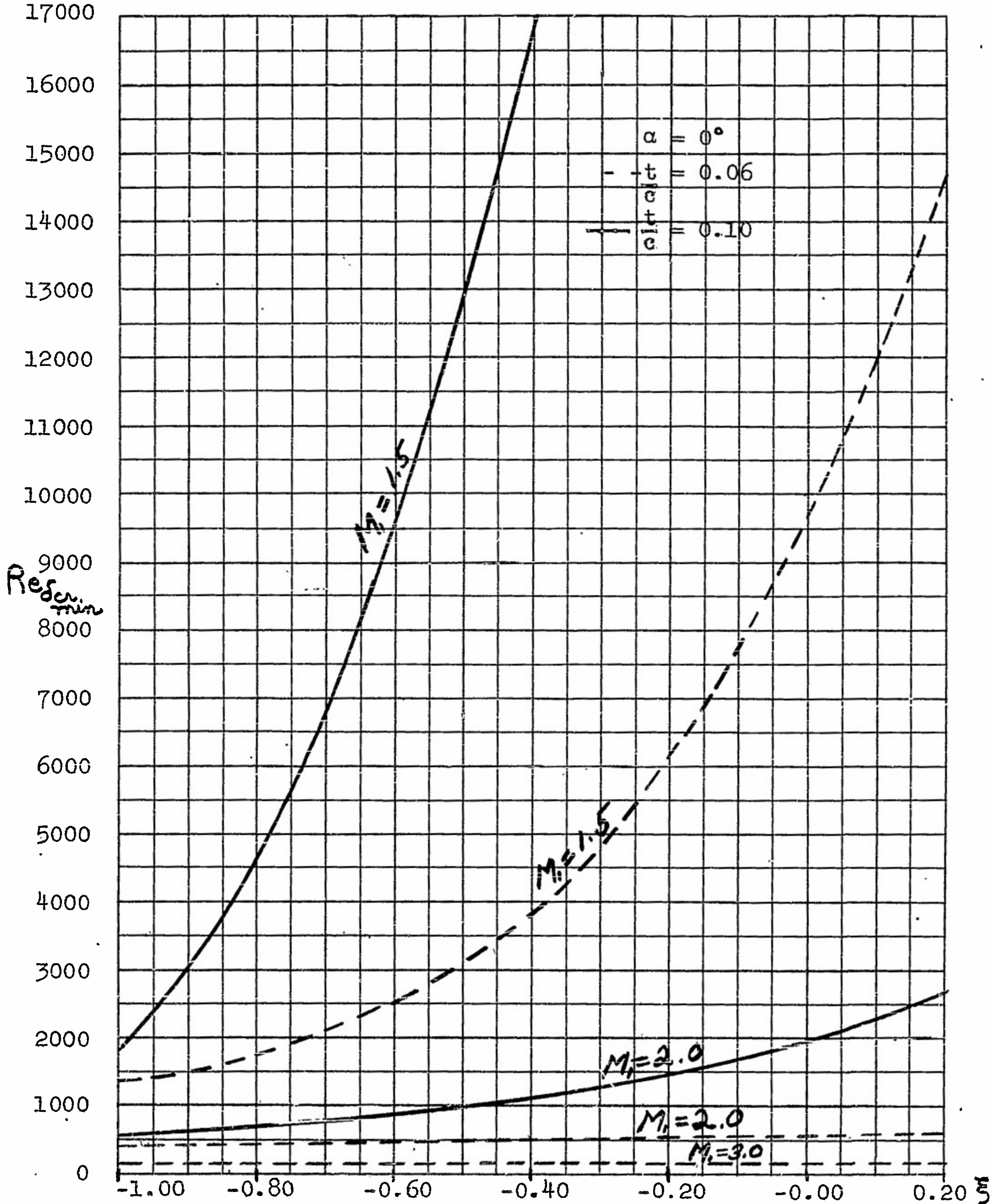

Figure 1a - Chordwise Distribution of Stability Limit, or Minimum Critical Reynolds Number $Re_{\delta_{cr, min}}$, For Symmetrical Circular-Arc Airfoil.

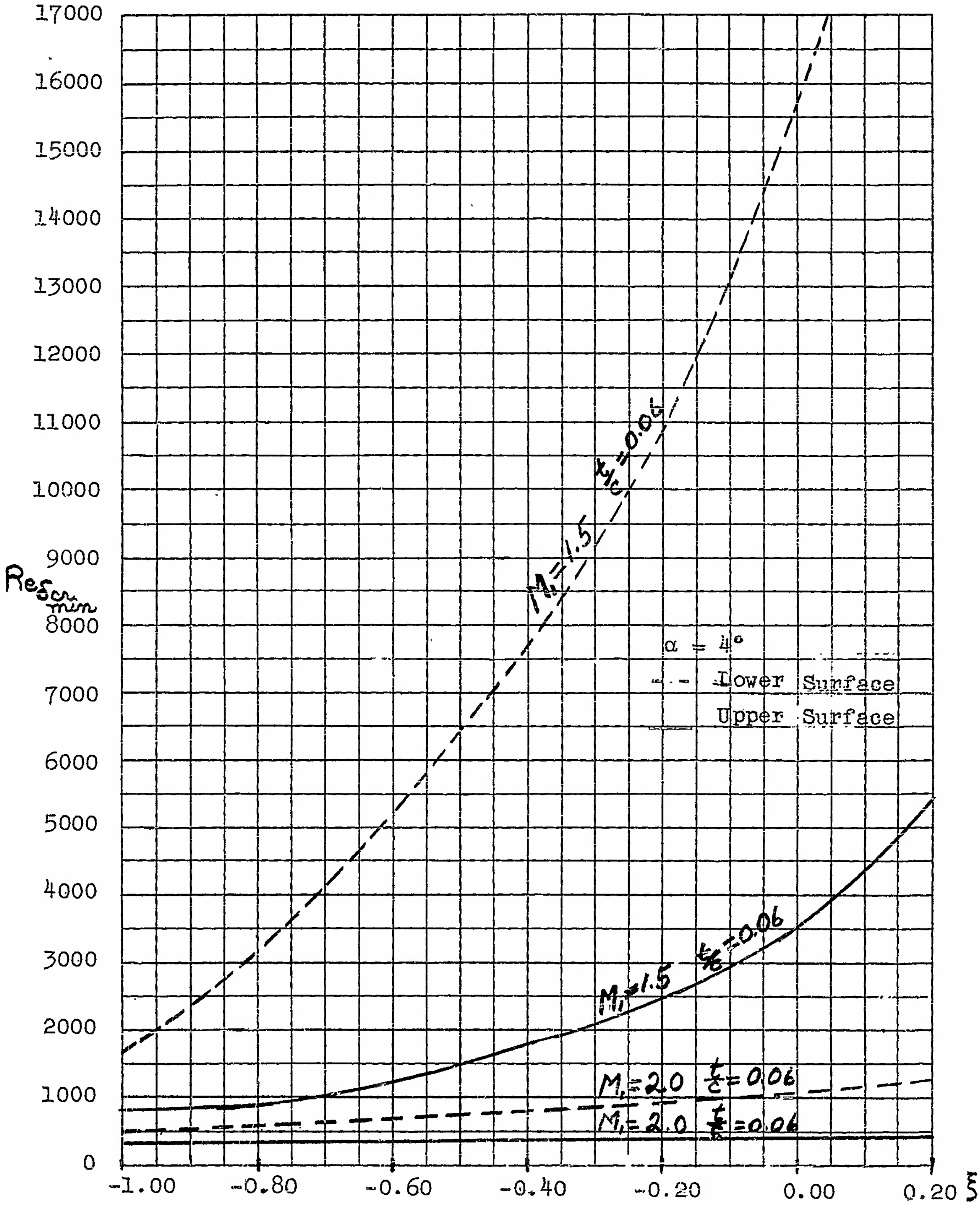

Figure 1b - Chordwise Distribution of Stability Limit, or Minimum Critical Reynolds Number $Re_{\delta_{cr\,min}}$, for Symmetrical Circular-Arc Airfoil.

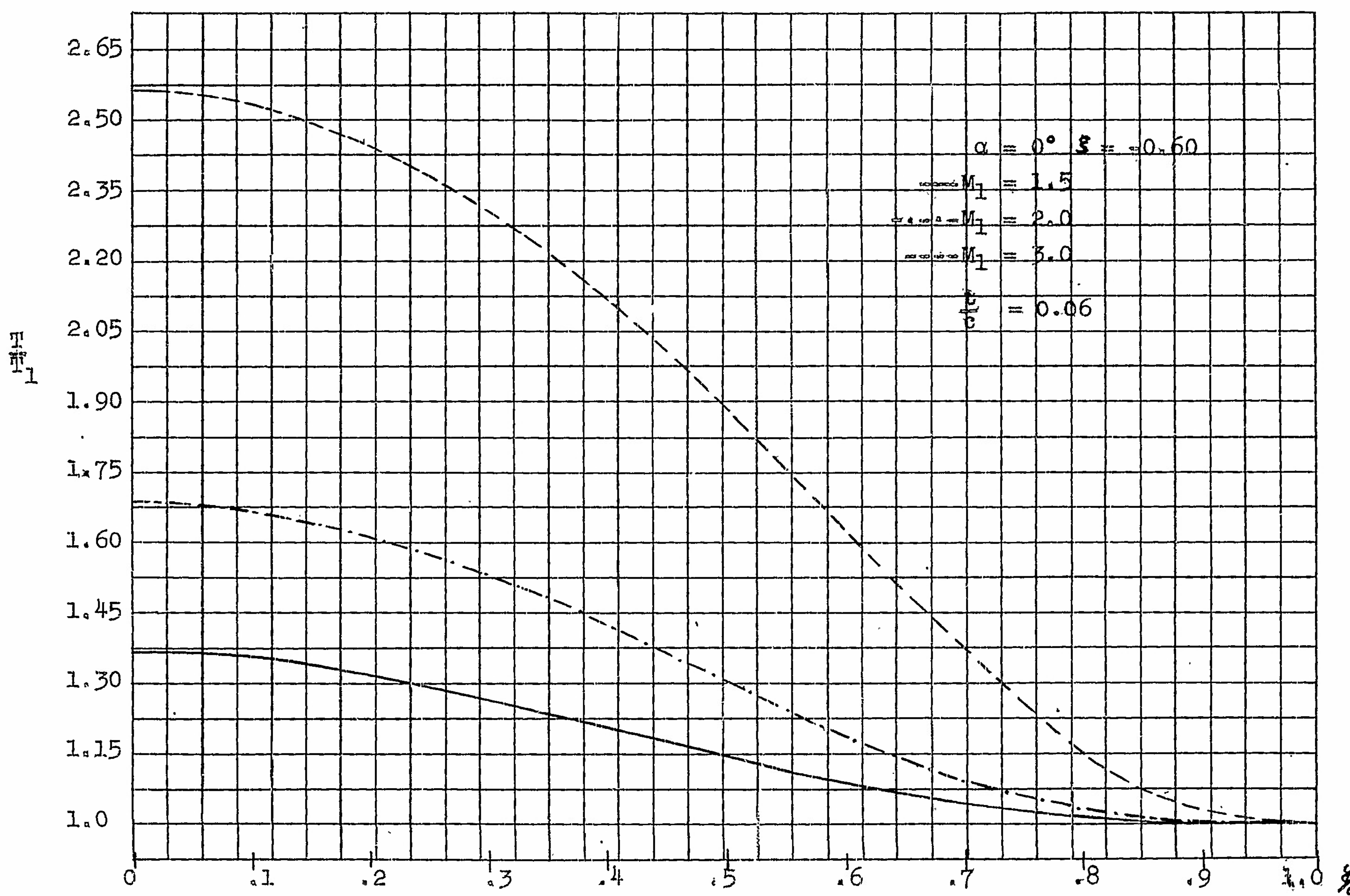

Figure 2 - Effect of Mach Number on Temperature Distribution Across Boundary Layer

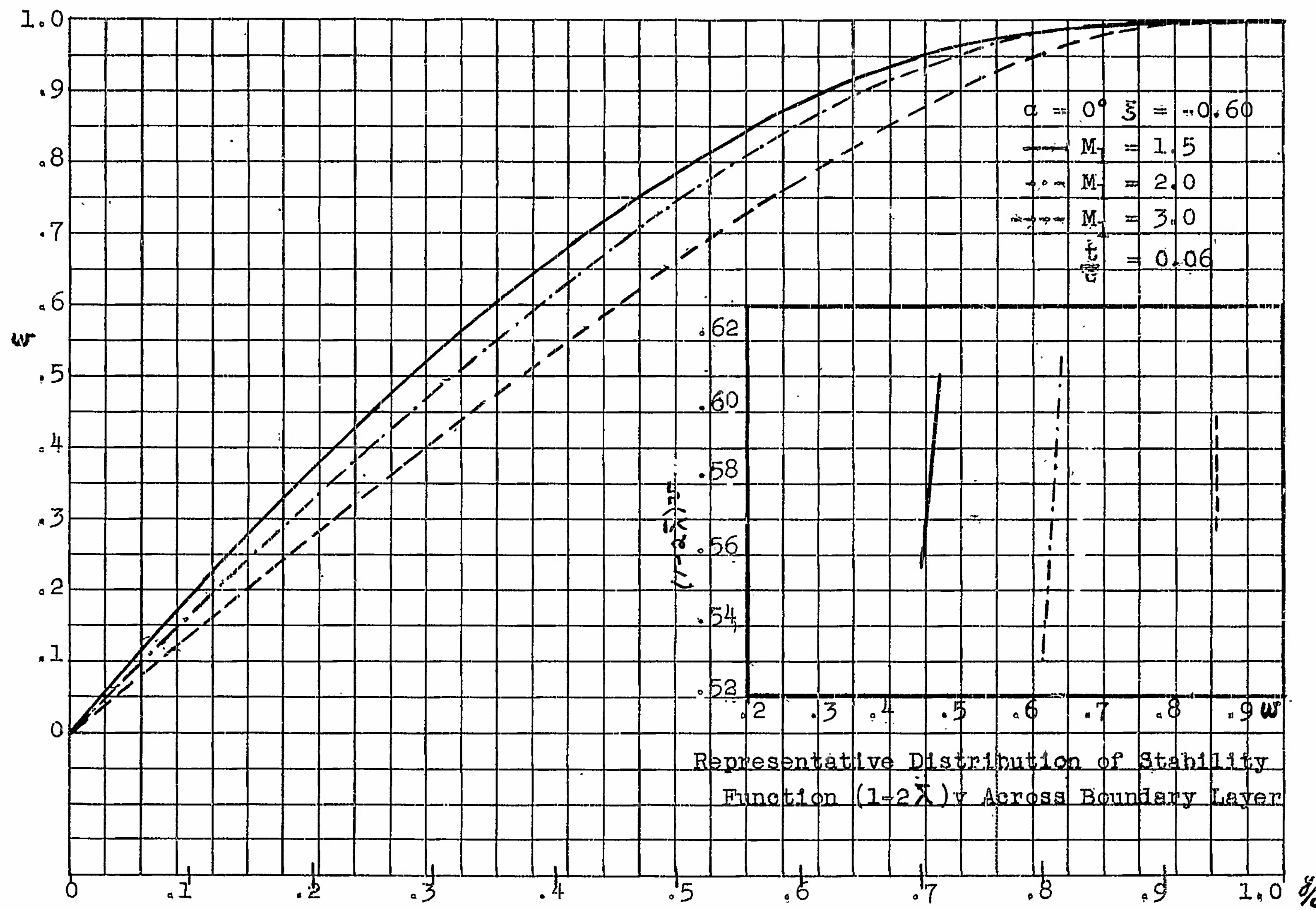

Figure 3 - Effect of Mach Number on Velocity Distribution Across Boundary Layer

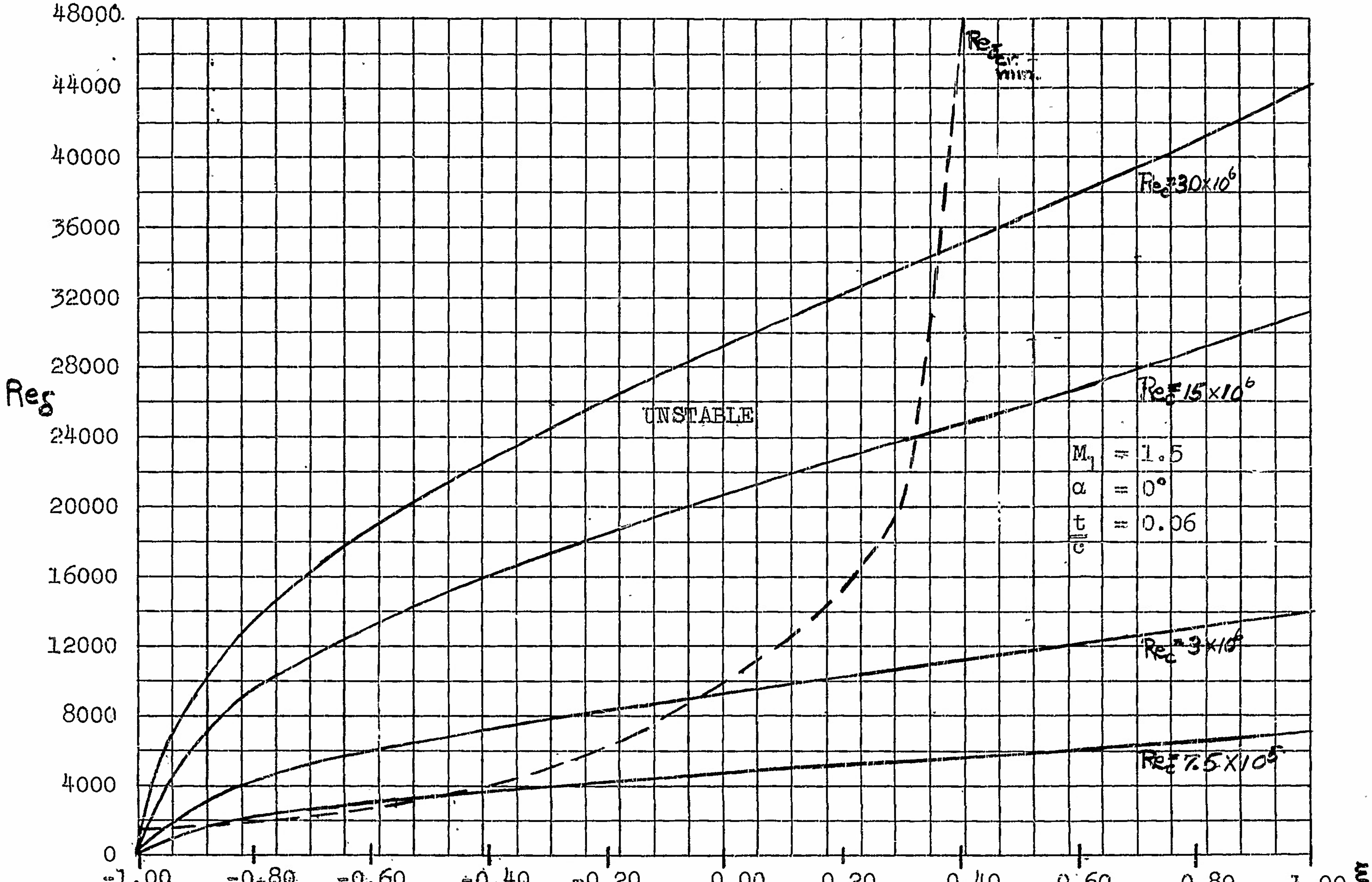

Figure 4a - Regions of Laminar Boundary Layer Instability for Symmetric Circular-Arc Airfoil

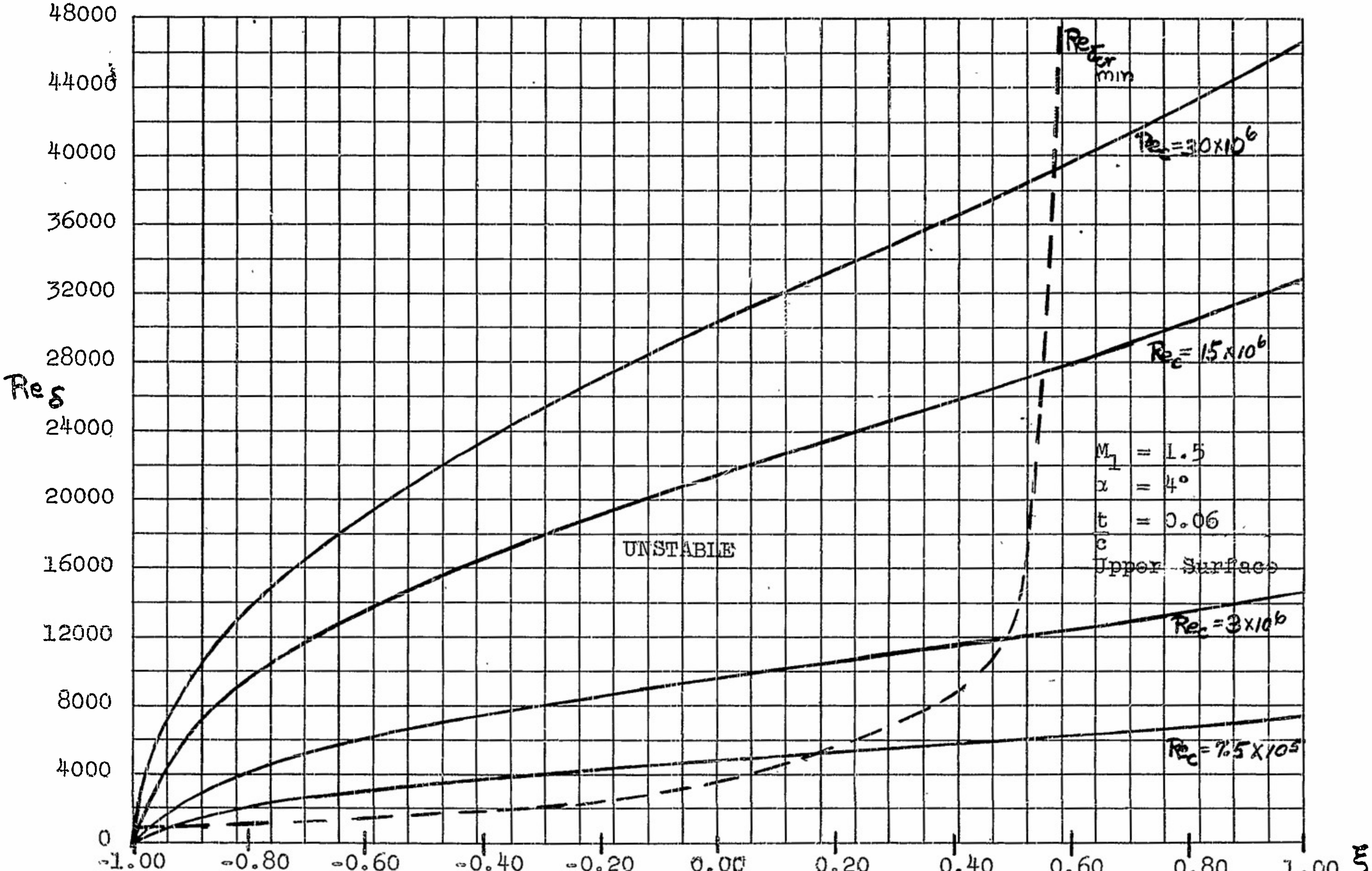

Figure 4b - Regions of Laminar Boundary Layer Instability for Symmetric Circular-Arc Airfoil

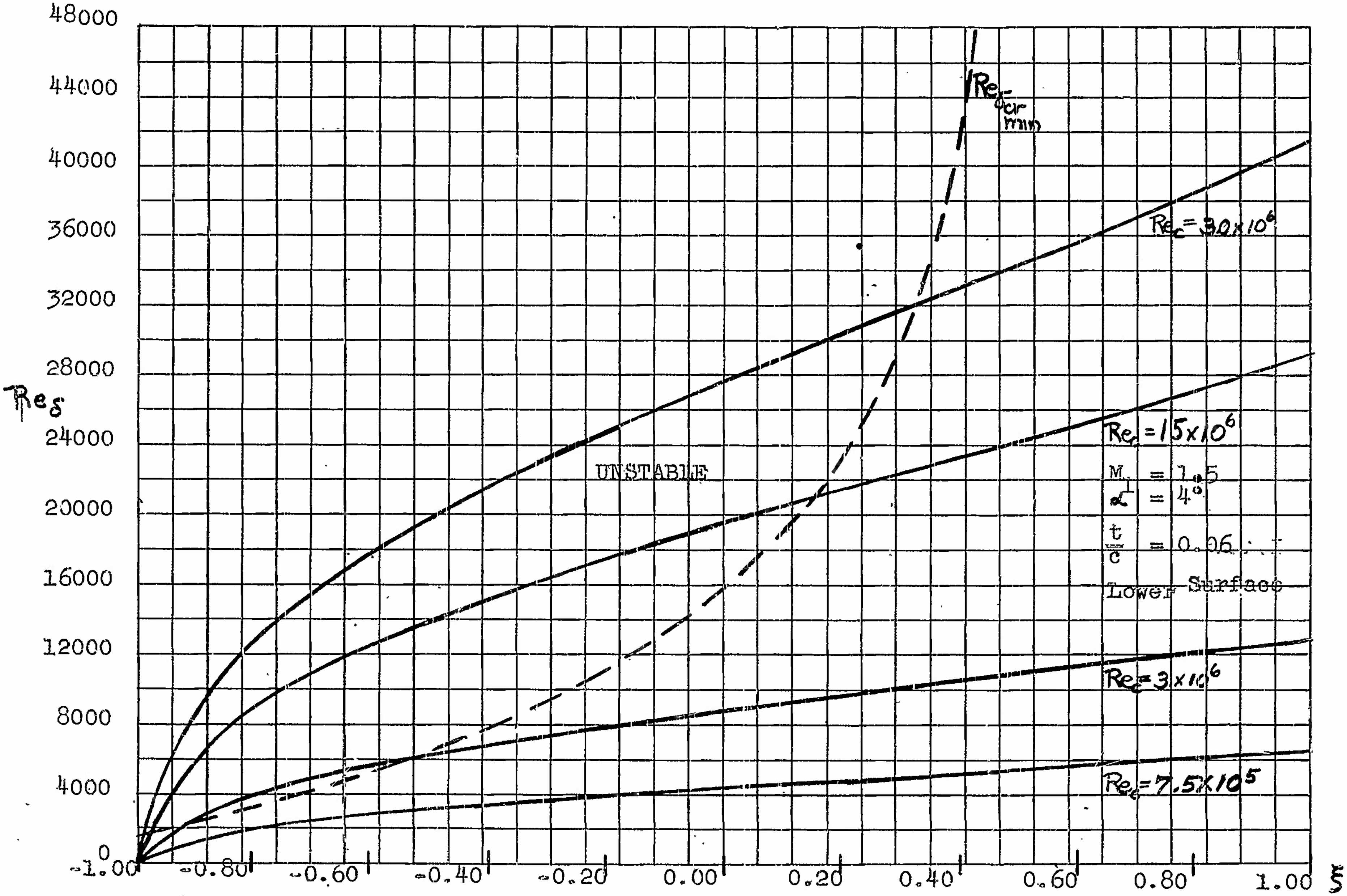

Figure 4c - Regions of Laminar Boundary Layer Instability for Symmetric Circular-arc Airfoil

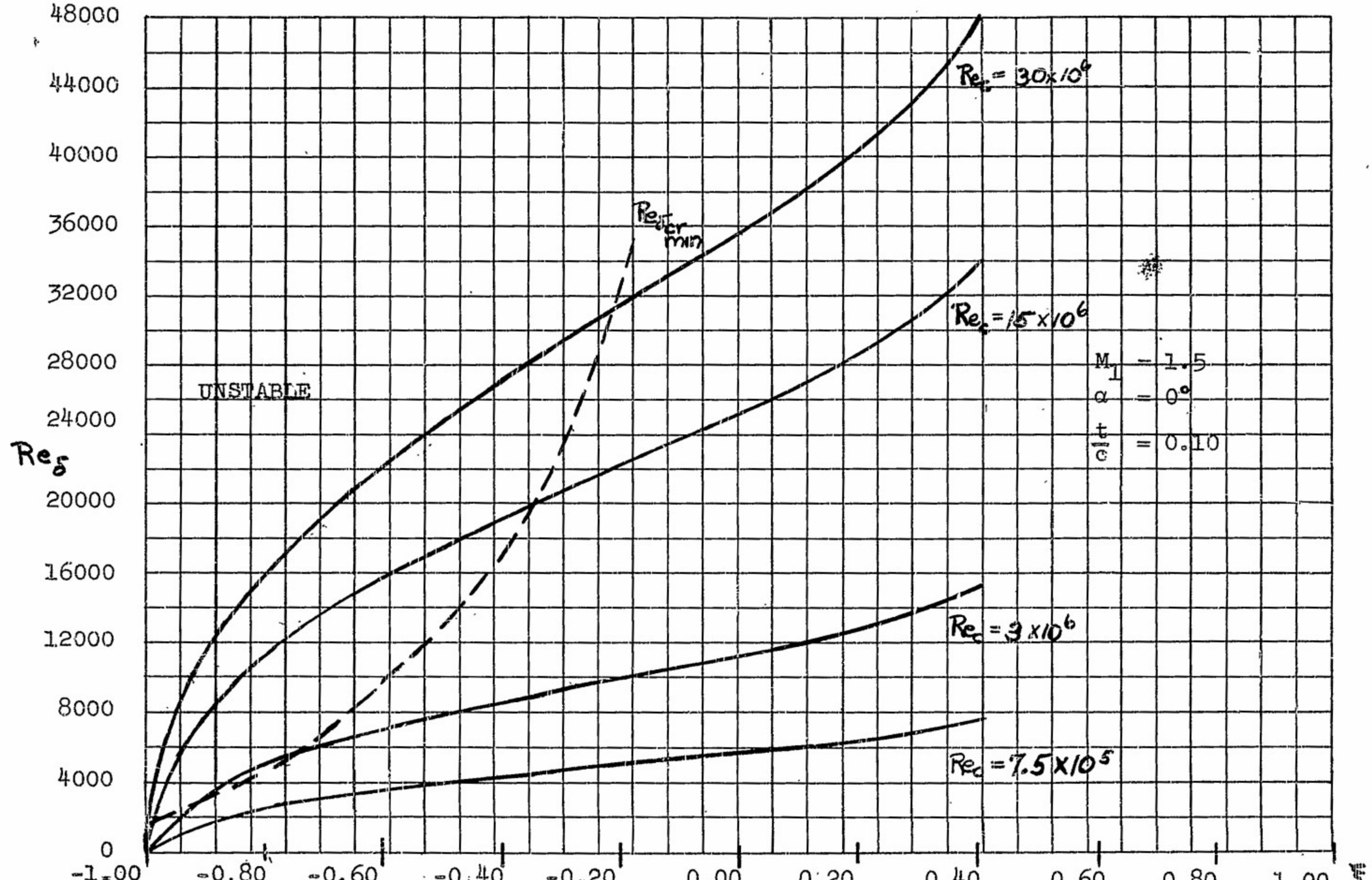

Figure 4d - Regions of Laminar Boundary Layer Instability for Symmetric Circular-Arc Airfoil

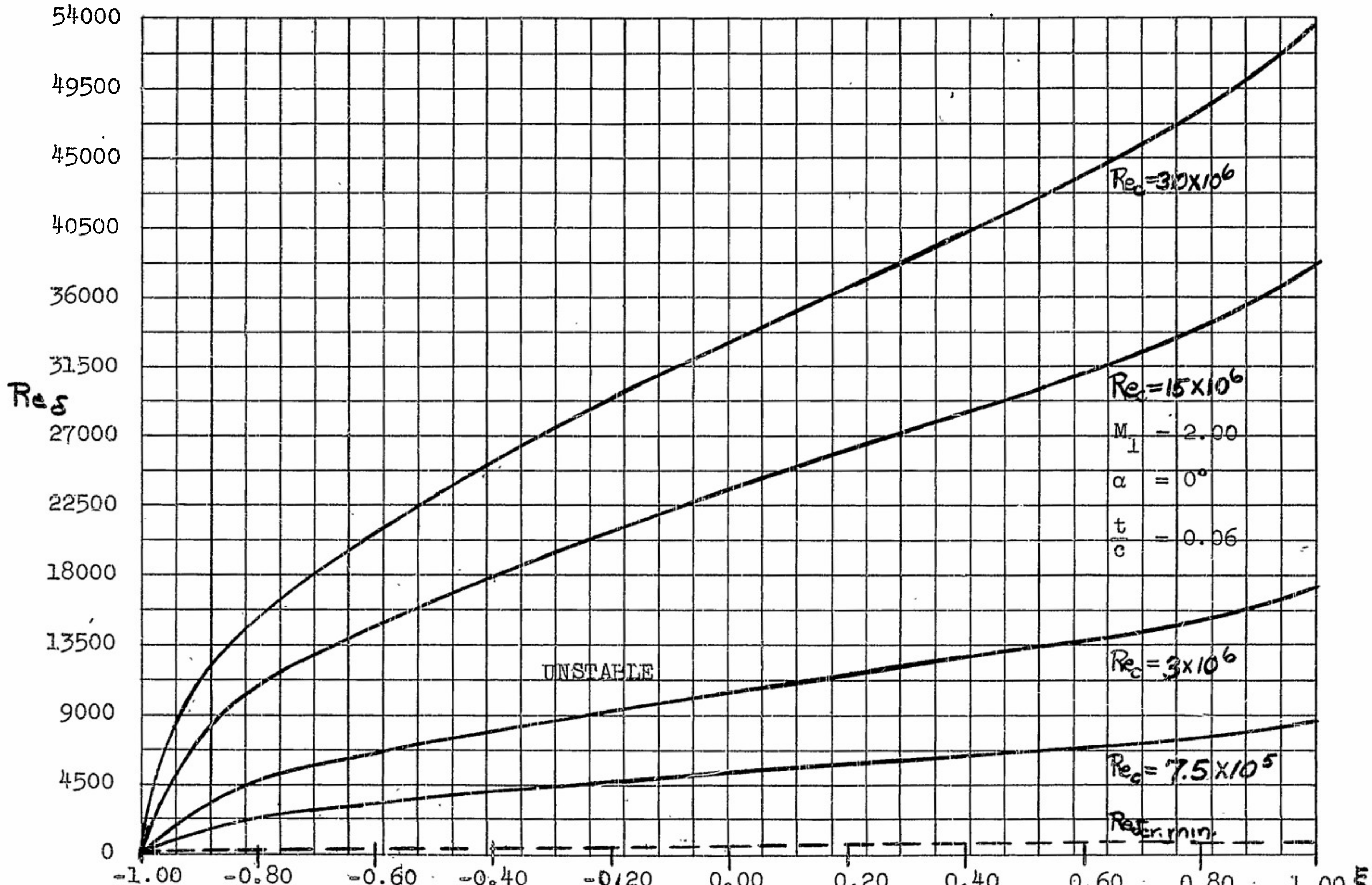

Figure 4e - Regions of Laminar Boundary-Layer Instability for Symmetric Circular-arc Airfoil

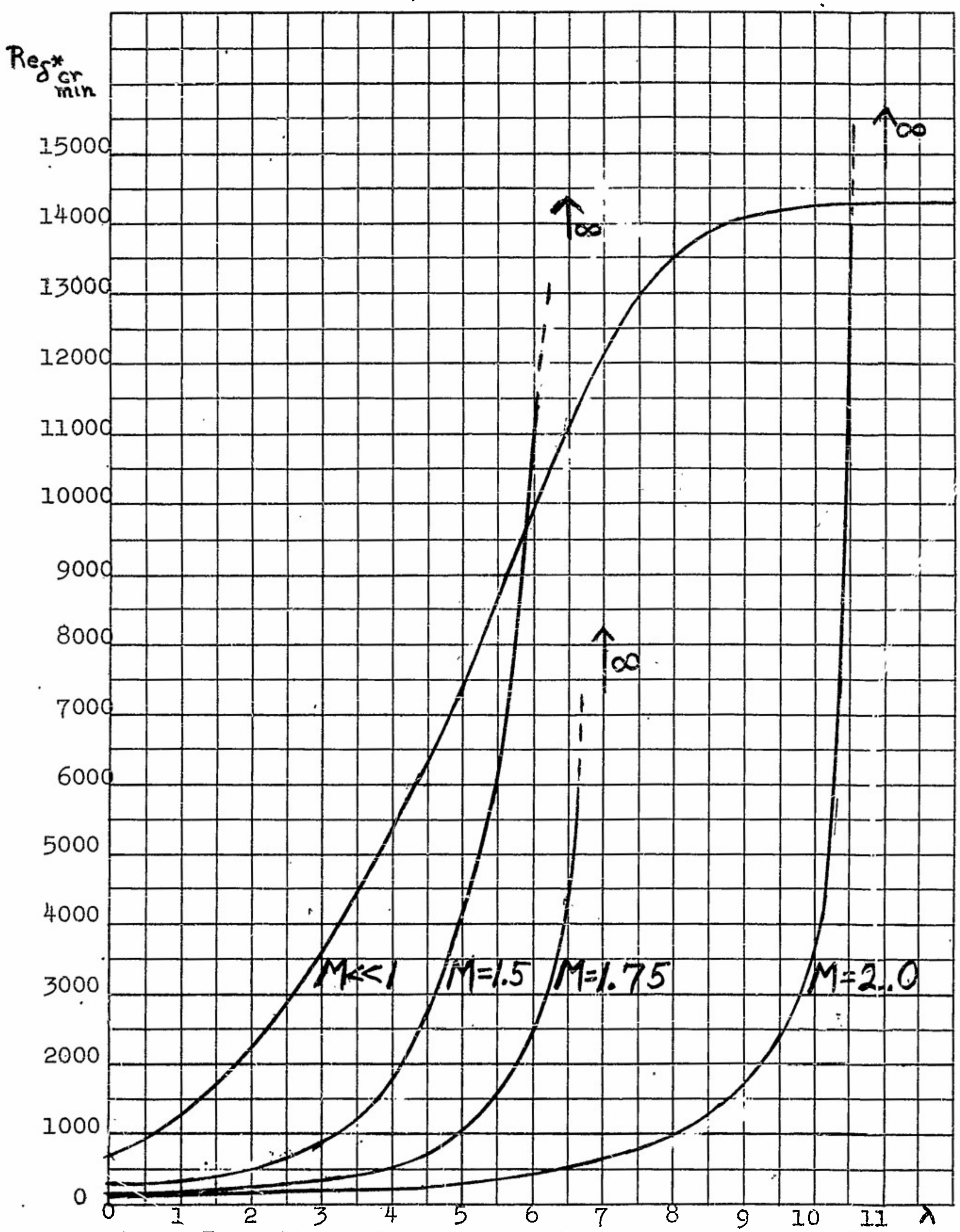

Figure 5 - Effect of Pressure Gradient on Limit of Stability for Laminar Boundary Layer Over an Insulated Surface.